Two-dimensional stationary Schrödinger equation via the $\bar{\alpha}$-dressing method: New exactly solvable potentials, wave functions, and their physical interpretation

V. G. Dubrovsky, a) A. V. Topovsky, and M. Yu. Basalaev

Novosibirsk State Technical University, Karl Marx prosp. 20, Novosibirsk 630092, Russia

(Received 23 June 2010; accepted 8 August 2010; published online 28 September 2010)

The classes of exactly solvable multiline soliton potentials and corresponding wave functions of two-dimensional stationary Schrödinger equation via $\bar{\alpha}$-dressing method are constructed and their physical interpretation is discussed. © 2010 American Institute of Physics. [doi:10.1063/1.3484162]

I. INTRODUCTION

Exact solutions of differential equations of mathematical physics, linear and nonlinear, are very important for the understanding of various physical phenomena. In the last three decades, the inverse scattering transform (IST) method has been generalized and successfully applied to several two-dimensional nonlinear evolution equations such as Kadomtsev–Petviashvili, Davey–Stewartson, Nizhnik–Veselov–Novikov, Zakharov–Manakov system, Ishimori, two-dimensional integrable Sin–Gordon, and others (see Refs. 1, 2, 24, and 25 and references therein).

The inverse scattering problem, i.e., recovering of potential and wave function from the scattering data, for the multidimensional stationary Schrödinger equation has been investigated in a number of papers. The one-dimensional case is well studied in works of Gelfand, Levitan, Feddeev, and Marchenko. But for higher dimension the difficulties because of overdetermination of inverse problem appear. In order to decrease this overdetermination, the inverse problem for fixed energy was stated. Important progress in study of multidimensional case has been achieved in the study of the two-dimensional Schrödinger equation both for the cases of periodic and rapidly decreasing potentials. An investigation of periodic case was initiated by Dubrovin et al. in the important paper. Therein, the problem of the exact integration of the two-dimensional Schrödinger equation $H\psi=E\psi$ with Hamiltonian, $H=-(\nabla^2-i\bar{\alpha}(x,y))^2+u(x,y)$, where $u$- and $\bar{\alpha}$-double periodic potentials (potentials of electric and magnetic fields), was posed and partially solved on the base of inverse data corresponding to one fixed energy level by the methods of the algebraic geometry. The “finite-zone with respect to a single energy level” 2D Schrödinger operators were constructed in Refs. 5–7. Here, also the conditions on the inverse problem data which lead to potential (i.e., without magnetic field $\bar{\alpha}=0$) and real Schrödinger operator ($u=\bar{u}$), has been found. Further development of this technique was received in Refs. 29 and 30.

The case of the potential decreasing at infinity was examined in works of Grinevich and Novikov R.G. 8,13; Grinevich and Novikov S.P. 9; Grinevich and Manakov 10,12; Grinevich 11,14,15; Novikov R.G. 16–18 (see also the books of Konopelchenko 24,25 and Ref. 26). In Ref. 8, the inverse problem for the 2D Schrödinger operator with magnetic fields,

$$L = -4\partial^2_{z\bar{z}} + A(z,\bar{z})\partial_z + B(z,\bar{z})\partial_{\bar{z}} + u(z,\bar{z}), \quad L\psi = E\psi,$$

on the basis of the nonlocal Riemann–Hilbert problem (Manakov 23) for its eigenfunction $\psi$ together with ideas from Refs. 6 and 7 was examined. The all coefficients of the operator were

a)Electronic addresses: dubrovsky@academ.org and dubrovsky@ngs.ru.
recovered by solution of the problem. In order to obtain potential \((A=B=0)\) and real \((u=\tilde{u})\) operator in Ref. 8 were formulated additional conditions (“reality” and “potentiality”) on the inverse problem data (kernel of the nonlocal Riemann–Hilbert problem). In accordance with these conditions, some class of such operators with negative eigenvalue (energy) has been constructed with theirs eigenfunctions. The potential \(u(x,y)\) in these operators is real and smooth function decreasing at infinity as \(r^{-1}, \; (r=\sqrt{x^2+y^2})\).

Another approach to reconstruction of the 2D Schrödinger operator with negative energy based on the \(\tilde{\phi}\)-problem\(^{21,22}\) has been realized in Refs. 9 and 12. Also, here the necessary conditions on the inverse problem data for the fast decaying at infinity potentials (decay faster than any inverse power of \(r\) when \(r\to\infty\)) in the case of negative energy were found.

The case of fixed positive energy was considered in Ref. 10 by combination of the nonlocal Riemann–Hilbert problem and the \(\tilde{\phi}\)-problem simultaneously. As well as in case of negative energy, the reality and potentiality condition on inverse problem data were formulated. Necessary conditions on the inverse problem data for the fast decaying at infinity potentials for positive energy were found in Refs. 13, 16, and 18.

The restrictions on inverse problem data which lead to transparent (i.e., scattering amplitude equals zero) at fixed positive energy potential are derived in Ref. 10. The explicit example of this potential, which is real, nonsingular, and decreasing rationally at infinity (as \(r^{-2}, r=\sqrt{x^2+y^2}\)) was constructed in Ref. 11. The construction of fast decaying potentials transparent at fixed energy and some its properties were examined in Ref. 13. Here it was shown that there are no nonzero real exponentially decreasing at infinity in all directions potentials transparent at a fixed energy, and there no nonzero real potentials transparent at an energy interval.

In Refs. 8–18, only the case of nonzero fixed energy was studied. The construction of the 2D Schrödinger operator with decreasing at infinity potentials with zero energy was examined in works of Boiti \textit{et al.}\(^9\) and Tsai.\(^20\)

A number of papers\(^{16–18}\) are devoted to the reconstruction of the 2D Schrödinger operator directly from the scattering amplitude-\(f\), which is determined from asymptotic representation of wave function for \(x^2+y^2\to\infty\),

\[
\psi(x,\bar{\kappa}) = e^{i(\bar{\kappa}x)} + f\left(\bar{\kappa}, \frac{x}{|x|}\right) e^{i|\bar{\kappa}|y} + o\left(\frac{1}{\sqrt{|x|}}\right), \quad \bar{x} = (x, y), \quad |\bar{\kappa}|^2 = E. \tag{2}
\]

The inverse problem data (which used in above-listed papers) in contrast to scattering amplitude \(f\) have no obvious physical meaning. Therefore, in Refs. 16–18, the connection between on the one hand the inverse problem data and on the other hand the scattering amplitude (for \(E>0\)) and additional scattering data (both for \(E>0\) and \(E<0\)) at a fixed energy is established.

In Ref. 18, the question of reconstruction of the 2D stationary Schrödinger operator with decreasing potentials directly via scattering amplitude \(f\) and additional scattering data was solved by the nonlocal Riemann–Hilbert problem and the \(\tilde{\phi}\)-equation for eigenfunctions. The uniqueness of this reconstruction at fixed energy was proven here. The conditions on scattering amplitude \(f\) and additional scattering data which lead to potential operator with real, bounded, and decreasing potential are formulated. The two cases of positive and negative energies were separately studied. In this article the reconstruction of the 2D Schrödinger operator based on nonlocal \(\tilde{\phi}\)-problem for its eigenfunction at fixed energy was also investigated. The conditions on the kernel of nonlocal \(\tilde{\phi}\)-problem corresponding to potential and real 2D Schrödinger operator were given.

In the present work the \(\tilde{\phi}\)-dressing method of Zakharov and Manakov\(^{25,27,28}\) is used for the construction of multilinear soliton potentials (in well known terminology one line soliton potential decreases exponentially only in one direction of the plane) and corresponding wave functions for 2D stationary Schrödinger equation,
Here $V_{Schr}(z, \bar{z})$ is scalar function, potential energy of a particle with unit mass $m=1$, $z=x+i\nu$, $\bar{z}=x-i\nu$, $\partial_z := \frac{1}{2}(\partial_x-i\partial_\nu)$, $\partial_{\bar{z}} := \frac{1}{2}(\partial_x+i\partial_\nu)$, $\partial_z := \frac{d}{d_t}$, $\partial_{\bar{z}} := \frac{d}{d_t}$, etc. The $\bar{\partial}$-dressing method is applied in the paper to linear problem,

$$L_1\psi = (\partial_{zz}^2 + u)\psi = 0,$$

with generically nonzero asymptotic value of $u(z, \bar{z})$ at infinity,

$$u(z, \bar{z}) = \tilde{u}(z, \bar{z}) + u_\infty = \tilde{u}(z, \bar{z}) - \epsilon, \quad \tilde{u}(z, \bar{z}) \mid_{|z|\to\infty} \to 0.$$  

Due to (3)–(5) the identification of potential energy $V_{Schr} := -2\tilde{u}(z, \bar{z})$ is valid and the $\bar{\partial}$-dressing is realized on fixed single level of energy $E := -2\epsilon$.

The $\bar{\partial}$-dressing method allows to construct integrable nonlinear evolution equations simultaneously with corresponding auxiliary linear problems and with enough fullness to investigate these problems. By this reason this method by its own right is also applicable for the analytic calculation of variable coefficients (exact potentials) and wave functions of linear problems without reference to nonlinear equations. This method is very convenient and one of the perspective method for solving the differential equations both nonlinear and linear ones. The use of $\bar{\partial}$-dressing method for the Schrödinger equation allows to construct different classes of real potentials with corresponding wave functions for different signs of energy in a unified and rather simple manner within the framework of this method.

Our paper is the continuation of Dubrovsky et al.\textsuperscript{26,31,32} works where the transparent potentials, one line soliton potentials (in Ref. 26), and potentials decreasing rationally at infinity with simple and multiple pole wave functions (in Refs. 31 and 32) have been calculated by $\bar{\partial}$-dressing method.

The paper is organized as follows. In Sec. II the basic ingredients of the $\bar{\partial}$-dressing method for 2D stationary Schrödinger equation (3) are presented, general determinant formulas for exactly solvable potentials $V_{Schr} = -2\tilde{u}$ and corresponding wave functions $\psi$ are derived. The fulfillment of potentiality condition for operator (4) in Sec. III is considered. The class of exactly solvable real multilinear soliton transparent potentials and corresponding wave functions of 2D stationary of Schrödinger equation in Sec. IV are constructed. The physical meaning of the simplest one line and two line soliton potentials and corresponding wave functions in Sec. V is discussed.

II. BASIC INGREDIENTS OF THE $\bar{\partial}$-DRESSING METHOD AND GENERAL DETERMINANT FORMULAS FOR EXACT SOLUTIONS

Here for convenience the basic ingredients of the $\bar{\partial}$-dressing method\textsuperscript{27,28} for 2D stationary Schrödinger equation (4) in the case of $u(z, \bar{z}) := \tilde{u}(z, \bar{z}) - \epsilon$ with generically nonzero asymptotic value $-\epsilon$ at infinity, i.e., $\tilde{u}(z, \bar{z}) \rightarrow 0$ as $|z| \rightarrow \infty$, are introduced in short.

At first one postulates the nonlocal $\bar{\partial}$-problem,

$$\frac{\partial \chi(\mu, \bar{\lambda})}{\partial \bar{\lambda}} = (\chi \ast R)(\lambda, \bar{\lambda}) = \int C \chi(\mu, \bar{\mu}) R(\mu, \bar{\mu}, \lambda, \bar{\lambda}) d\mu \wedge d\bar{\mu},$$

where in our case $\chi$ and $R$ are the scalar complex-valued functions and $\chi$ has canonical normalization: $\chi \rightarrow 1$ as $\lambda \rightarrow \infty$. We also assume that problem (6) is unique solvable. Then one introduces the dependence of kernel $R$ of $\bar{\partial}$-problem (6) on the space variables $z, \bar{z}$,

$$\frac{\partial R}{\partial z} = i\mu R(\mu, \bar{\mu}, \lambda, \bar{\lambda}; z, \bar{z}) - R(\mu, \bar{\mu}, \lambda, \bar{\lambda}; z, \bar{z}) i\lambda.$$
\[
\frac{\partial R}{\partial \bar{z}} = -i\frac{\epsilon}{\mu} R(\mu, \tilde{\mu}; \lambda, \lambda \bar{z}, \bar{z}) + R(\mu, \tilde{\mu}; \lambda, \lambda \bar{z}, \bar{z})i \frac{\epsilon}{\lambda}.
\] (7)

Integrating (7), one obtains
\[
R(\mu, \tilde{\mu}; \lambda, \lambda \bar{z}, \bar{z}) = R_0(\mu, \tilde{\mu}; \lambda, \lambda) e^{F(\mu, \tilde{\mu}; \lambda, \lambda \bar{z}, \bar{z})}, \quad F(\lambda; z, \bar{z}) := i \left[ \lambda z - \frac{\epsilon}{\lambda} \right].
\] (8)

By the use of “long” derivatives \( D_1 = \partial_\lambda + i\lambda \), \( D_2 = \partial_\bar{z} - i\bar{z} \), expressing dependence (7) of kernel \( R \) of \( \bar{\partial} \)-problem (6) on the space variables \( z, \bar{z} \) in the following equivalent form \([D_1, R] = 0, [D_2, R] = 0\), one can construct the operator of auxiliary linear problem,
\[
\bar{L} = \sum_{l,m} u_{lm}(z, \bar{z}) D_1^l D_2^m.
\] (9)

This operator must satisfy the conditions
\[
\left[ \frac{\partial}{\partial \lambda}, \bar{L} \right] \chi = 0, \quad \bar{L} \chi(\lambda, \bar{\lambda})|_{\lambda=\infty} \to 0
\] (10)
of absence singularities at the points \( \lambda=0 \) and \( \lambda=\infty \) of the complex plane of spectral variable \( \lambda \). For such operators \( \bar{L} \), the function \( \bar{L} \chi \) obeys the same \( \bar{\partial} \)-equation as the function \( \chi \). There may be several operators \( \bar{L} \) of this type, by virtue of the unique solvability of (6), one has \( \bar{L} \chi = 0 \) for each of them. In considered case, one constructs the operator
\[
\bar{L}_1 = D_1 D_2 + u_1 D_1 + u_2 D_2 + u.
\] (11)

Using conditions (10) and series expansions of wave functions \( \chi \) near the points \( \lambda=0 \) and \( \lambda=\infty \),
\[
\chi = \chi_0 + \chi_1 \lambda + \chi_2 \lambda^2 + \ldots, \quad \chi = \chi_\infty + \frac{\chi_{-1}}{\lambda} + \frac{\chi_0}{\lambda^2} + \ldots,
\] (12)
one obtains the reconstruction formulas for the field variables \( u_1, u_2 \) through the coefficients \( \chi_0 \) and \( \chi_\infty \),
\[
u_1 = -\chi_{-2}/\chi_\infty, \quad u_2 = -\chi_0/\chi_0.
\] (13)

In well known terminology, the operator \( \bar{L}_1 \) in (11) is a pure potential operator if the terms with the first derivatives in it are absent. Due to canonical normalization of wave function \( \chi|_{\lambda=\infty} = 1 \) and
\[
u_1 = -\chi_{-2}/\chi_\infty = 0.
\] (14)

For zero value of the term \( u_2 \partial_\bar{z} \) in \( \bar{L}_1 \), one must require \( \chi_0 = \text{const} \), without loss of generality, we can choose \( \chi_0 = 1 \) and then
\[
u_2 = -\chi_0/\chi_0 = 0.
\] (15)

Using (10) and (13)–(15), one obtains the following expressions for \( u \):
\[
u = -\epsilon - i\epsilon \chi_{-1} - \epsilon + i\epsilon \chi_1.
\] (16)

In terms of the wave function
\[ \psi = \chi e^{F(\lambda; z, \bar{z})} = \chi e^{i(\lambda \bar{z} - e\lambda z)}, \]  

under the reduction \( u_1 = 0 \) and \( u_2 = 0 \) (the condition of potentiality \( \bar{L}_1 \)), one obtains from (11) due to (10) and (14)–(17) linear problem (4).

The solution of \( \bar{\delta} \)-problem (6) with constant normalization \( \chi_0 = 1 \) is equivalent to the solution of the following singular integral equation:

\[ \chi(\lambda) = 1 + \int \int \frac{d\lambda' \wedge d\overline{\lambda}'}{2\pi i(\lambda' - \lambda)} \int \int \chi(\mu, \bar{\mu}) R(\mu, \bar{\mu}; \lambda', \bar{\lambda'}) d\mu \wedge d\bar{\mu}. \]  

From (18), one obtains for the coefficients \( \chi_0 \) and \( \chi_{-1} \) of series expansions (12) of \( \chi \) the following expressions:

\[ \chi_0 = 1 + \int \int \frac{d\lambda \wedge d\bar{\lambda}}{2\pi i \lambda} \int \int \chi(\mu, \bar{\mu}) R_0(\mu, \bar{\mu}; \lambda, \bar{\lambda}) e^{F(\mu) - F(\lambda)} d\mu \wedge d\bar{\mu} \]  

and

\[ \chi_{-1} = - \int \int \frac{d\lambda \wedge d\bar{\lambda}}{2\pi i \lambda} \int \int \chi(\mu, \bar{\mu}) R_0(\mu, \bar{\mu}; \lambda, \bar{\lambda}) e^{F(\mu) - F(\lambda)} d\mu \wedge d\bar{\mu}, \]  

where \( F(\lambda) \) is short notation for \( F(\lambda; z, \bar{z}) \) given by formula (8). The conditions of reality \( u \) and of potentiality of the operator \( L_1 \) give some restrictions on the kernel \( R_0 \) of \( \bar{\delta} \)-problem (6). For considered case with complex space variables \( z = x + iy, \bar{z} = x - iy \) the condition of reality of \( u \) leads from (16) and (20) in the limit of “weak” fields to the following restriction on the kernel \( R_0 \) of the \( \bar{\delta} \)-problem:

\[ R_0(\mu, \bar{\mu}; \lambda, \bar{\lambda}) = \frac{\epsilon^3}{|\mu|^2|\lambda|^2 \mu \bar{\lambda}} R_0 \left( -\frac{\epsilon}{\mu}, \frac{\epsilon}{\bar{\lambda}}, \frac{\epsilon}{\bar{\mu}}, \frac{\epsilon}{\mu} \right). \]  

The potentiality condition for the operator \( \bar{L}_1 \) in (11) for the choice \( \chi_0 = 1 \) due to (19) has the following form:

\[ \chi_0 - 1 = \int \int \frac{d\lambda \wedge d\bar{\lambda}}{2\pi i \lambda} \int \int \chi(\mu, \bar{\mu}) R_0(\mu, \bar{\mu}; \lambda, \bar{\lambda}) e^{F(\mu) - F(\lambda)} d\mu \wedge d\bar{\mu} = 0. \]  

It will be useful to obtain general formulas for exactly solvable potentials \( u(z, \bar{z}) \) corresponding to the degenerate delta-kernel \( R_0 \):

\[ R_0(\mu, \bar{\mu}; \lambda, \bar{\lambda}) = \pi \sum_k A_k \delta(\mu - M_k) \delta(\lambda - \Lambda_k). \]  

In this case the wave function \( \chi(\lambda) \) due to (18) has the form

\[ \chi(\lambda) = 1 + 2i \sum_k \frac{A_k}{\Lambda_k - \lambda} \chi(M_k) e^{F(M_k) - F(\Lambda_k)} \]  

The coefficient \( \chi_{-1} \) due to (20) and (23) is given by expression

\[ \chi_{-1} = - 2i \sum_k A_k \chi(M_k) e^{F(M_k) - F(\Lambda_k)}. \]  

For the wave functions \( \chi(M_k) \) from (24), one obtains the following system of equations:
\[ \sum_k \tilde{A}_{ik}(M_i) = 1, \quad \tilde{A}_{ik} = \delta_{ik} + \frac{2iA_k}{M_i - \Lambda_k} e^{F(M_i) - F(\Lambda_i)}. \] (26)

It is convenient instead of matrix \( \tilde{A} \) in (26) to introduce the matrix \( A \) given by expression

\[ A_{ik} := e^{F(M_i)} \tilde{A}_{ik} e^{-F(M_i)} = \delta_{ik} + \frac{2iA_k}{M_i - \Lambda_k} e^{F(M_i) - F(\Lambda_i)}. \] (27)

From (26) due to (27), one derives the expression for the wave function \( \chi \) at discrete values of spectral variable,

\[ \chi(M_i) = \sum_k \tilde{A}^{-1}_{ik} = \sum_k e^{F(M_i) - F(\Lambda_i)} A^{-1}_{ik}. \] (28)

It will be useful to give some formulas for wave functions satisfying to linear auxiliary problems (4). From (17) and (28), one obtains the wave function \( \psi(M_i, z, \bar{z}) = \chi(M_i) e^{F(M_i)} \) at discrete points \( \lambda = M_i \) in the space of spectral variables,

\[ \psi(M_i, z, \bar{z}) = \chi(M_i) e^{F(M_i)} = \sum_k e^{F(M_i)} A^{-1}_{ik}. \] (29)

For wave function (17) at continuous value of spectral variable \( \lambda \), from (24)–(28) follows the expression

\[ \psi(\lambda, z, \bar{z}) = \chi(\lambda) e^{F(\lambda)} = \left[ 1 + 2i \sum_k \frac{A_k}{\Lambda_k - \lambda} e^{F(M_i) - F(\Lambda_i)} \chi(M_k) \right] e^{F(\lambda)} = \left[ 1 + 2i \sum_k \frac{A_k}{\Lambda_k - \lambda} e^{F(\Lambda_i)} A^{-1}_{ki} e^{F(M_i)} \right] e^{F(\lambda)}. \] (30)

Inserting (28) into (25), one obtains for the coefficient \( \chi_{-1} \)

\[ \chi_{-1} = -2i \sum_{k,j} A_{k} e^{F(M_i) - F(\Lambda_j)} A^{-1}_{kj} = i \text{tr} \left( \frac{\partial A}{\partial \bar{z}} A^{-1} \right), \] (31)

and due to reconstruction formula \( u = -\epsilon - i\chi_{-1} \bar{z} \), the convenient determinant formula for the potentials \( u \) of linear problem (4),

\[ u = -\epsilon + \frac{\partial}{\partial \bar{z}} \left( \frac{\partial A}{\partial \bar{z}} A^{-1} \right) = -\epsilon + \frac{\partial^2}{\partial \bar{z} \partial \bar{z}} \ln(\det A). \] (32)

Here and below, useful determinant identities,

\[ \text{Tr} \left( \frac{\partial A}{\partial \bar{z}} A^{-1} \right) = \frac{\partial}{\partial \bar{z}} \ln(\det A), \quad 1 + \text{tr} D = \det(1 + D), \] (33)

are used; the matrix \( D \) from last identity of (33) is degenerate with rank 1.

Potentiality condition (22) by the use of (24)–(28) can be transformed to the form

\[ \chi_0 - 1 = -\frac{1}{2\epsilon} \sum_{k,l=1}^N A_{kl}^1 B_{kl} = 0, \quad B_{kl} := -\frac{4i\epsilon}{\Lambda_k} A_k e^{F(M_i) - F(\Lambda_i)}, \] (34)

where degenerate matrix \( B \) has rank 1. Due to (22) and to (33) and (34), potentiality condition (34) takes the form
\[ 0 = \sum_{k,n=1}^{N} A_{km}^{-1} B_{mk} = \text{tr}(A^{-1}B) = \det(BA^{-1} + 1) - 1 = 0 \Rightarrow \det(A + B) = \det A, \] (35)

here matrix \( BA^{-1} \) is degenerate of rank 1 and in deriving (35) the second matrix identity of (33) is used.

**III. FULFILMENT OF POTENTIALITY CONDITION. GENERAL FORMULAS FOR THE SIMPLEST EXACT POTENTIALS AND CORRESPONDING WAVE FUNCTIONS**

Formula (32) for exact potentials \( u(z, \bar{z}) \) of linear problem (4) will be effective if the reality \( \bar{u}=u \) and potentiality conditions (21) and (35) of operator \( L_1 \) are satisfied. It will be shown at first how one can fulfill the condition of potentiality (22) or (35) by delta-kernel \( R_0 \) with two terms,

\[ R_0(\mu, \bar{\mu}; \lambda, \bar{\lambda}) = \pi(A \delta(\mu - \mu_1) \delta(\lambda - \lambda_1) + B \delta(\mu - \mu_2) \delta(\lambda - \lambda_2)). \] (36)

Inserting (36) into (22), one obtains in the limit of weak \( \chi=1 \) in (22) fields,

\[ \chi_0 - 1 = \left( \int_C \frac{1}{2i\lambda} \left( A \delta(\mu - \mu_1) \delta(\lambda - \lambda_1) + B \delta(\mu - \mu_2) \delta(\lambda - \lambda_2) \right) \right. \]
\[ \left. \times e^{F(\mu) - F(\lambda)} d\mu \wedge d\bar{\mu} \wedge d\lambda \wedge d\bar{\lambda} = 2 \left( \frac{A}{\lambda_1} e^{F(\mu_1) - F(\lambda_1)} + \frac{B}{\lambda_2} e^{F(\mu_2) - F(\lambda_2)} \right) \right) = 0. \] (37)

Equality (37) is valid if

\[ F(\mu_1) - F(\lambda_1) = F(\mu_2) - F(\lambda_2), \quad \frac{A}{\lambda_1} = -\frac{B}{\lambda_2}. \] (38)

Due to the definition of \( F(\lambda)=i[\lambda z - \frac{\bar{z}}{\lambda}] \) from (38) follows the system of equations,

\[ \mu_1 - \lambda_1 = \mu_2 - \lambda_2, \quad \frac{e}{\mu_1} - \frac{e}{\lambda_1} = \frac{e}{\mu_2} - \frac{e}{\lambda_2}. \] (39)

System (39) has the following solutions:

(1) \( \mu_1 = \lambda_1, \quad \mu_2 = \lambda_2 \), (2) \( \mu_1 = -\lambda_2, \quad \mu_2 = -\lambda_1 \).

(40)

The solution \( \mu_1 = \lambda_1, \quad \mu_2 = \lambda_2 \) corresponds to lump solution and will not be considered here, see about lump solutions Refs. 31 and 32. For the second solution \( \mu_1 = -\lambda_2, \quad \mu_2 = -\lambda_1 \) taking into account (38), one obtains the relation

\[ A/\lambda_1 = -B/\lambda_2 = B/\mu_1 = a, \] (41)

here \( a \) is some arbitrary complex constant. Finally, one concludes that to potentiality condition (22) satisfies the kernel \( R_0 \) which is the sum of expressions of type (36) with parameters defined by (39)–(41),

\[ R_0(\mu, \bar{\mu}; \lambda, \bar{\lambda}) = \pi \sum_{k=1}^{N} \left[ a_k \lambda_k \delta(\mu - \mu_k) \delta(\lambda - \lambda_k) + a_k \mu_k \delta(\mu + \lambda_k) \delta(\lambda + \mu_k) \right] \]
\[ = \pi \sum_{k=1}^{2N} A_k \delta(M - M_k) \delta(\Lambda - \Lambda_k), \] (42)

where the sets of amplitudes \( A_k \) and spectral parameters \( M_k, \Lambda_k \) are given by expressions

\[ (A_1, \ldots, A_{2N}) := (a_1 \lambda_1, \ldots, a_N \lambda_N; a_1 \mu_1, \ldots, a_N \mu_N). \]
\[ (M_1, \ldots, M_{2N}) := (\mu_1, \ldots, \mu_N; -\lambda_1, \ldots, -\lambda_N), \]

\[ (\Lambda_1, \ldots, \Lambda_{2N}) := (\lambda_1, \ldots, \lambda_N; -\mu_1, \ldots, -\mu_N). \]  

In order to avoid repetition of similar calculations in the succeeding sections, it will be convenient to prepare some useful formulas in general position for calculating exact one and two line soliton potentials \( u \) and corresponding wave functions. The determinants of matrix \( A \) (27) with parameters (43) corresponding to simplest kernels (42) with \( N=1 \) and \( N=2 \) have the forms

1. \( N=1: \det A = (1 + p_1 e^{\Delta F(\mu; \lambda)})^2, \)  

2. \( N=2: \det A = (1 + p_1 e^{\Delta F(\mu; \lambda)} + p_2 e^{\Delta F(\mu, \lambda)} + q e^{\Delta F(\mu; \lambda) + \Delta F(\mu, \lambda)})^2, \)

where \( p_k, \Delta F(\mu, \lambda) \) \((k=1,2)\), and \( q \) by the expressions

\[ p_k := ia \frac{\mu_k + \lambda_k}{\mu_k - \lambda_k}; \quad \Delta F(\mu_k, \lambda_k) := F(\mu_k) - F(\lambda_k), \]

\[ q := -p_1 p_2 \left( \frac{(\lambda_1 - \lambda_2)(\lambda_2 + \mu_1)(\mu_1 - \mu_2)(\lambda_1 + \mu_2)}{(\lambda_1 + \lambda_2)(\lambda_2 - \mu_1)(\mu_1 + \mu_2)(\lambda_1 - \mu_2)} \right) \]

are given. The formula for simplest \([\text{corresponding to one term } N=1 \text{ in } R_0 (42)]\) exact potential \( u_{N=1} \) due to (32) and (44) has the form

\[ u_{N=1}(z, \bar{z}) = -\epsilon - \epsilon \frac{2p_1(\mu_1 - \lambda_1)^2}{\mu_1 \lambda_1} \frac{e^{\Delta F(\mu_1; \lambda_1)}}{(1 + p_1 e^{\Delta F(\mu_1; \lambda_1)})^2}. \]  

By the use of (24), (27), and (28) corresponding to potential (48) wave functions,

\[ \tilde{\chi}_1 := \chi_1(\mu_1) = \chi_1(-\lambda_1) = \frac{1}{1 + p_1 e^{\Delta F(\mu_1; \lambda_1)}}, \]

\[ \chi_1(\lambda) = 1 - \left( \frac{\lambda_1}{\lambda - \lambda_1} + \frac{\mu_1}{\lambda + \mu_1} \right) \frac{2i a e^{\Delta F(\mu_1; \lambda_1)}}{1 + p_1 e^{\Delta F(\mu_1; \lambda_1)}} \]

one calculates. Corresponding to (49) and (50) wave functions \( \psi \) at discrete values of spectral parameters \( \psi_1(\mu_1) = \chi_1(\mu_1) e^{F(\mu_1)}, \quad \psi_1(-\lambda_1) = \chi_1(-\lambda_1) e^{F(-\lambda_1)}, \) and at continuous value of spectral parameter \( \psi_1(\lambda) = \chi_1(\lambda) e^{F(\lambda)} \) satisfying to linear auxiliary problem (4) have the following forms:

\[ \psi_1(\mu_1) = \frac{e^{F(\mu_1)}}{1 + p_1 e^{\Delta F(\mu_1; \lambda_1)}}, \quad \psi_1(-\lambda_1) = \frac{e^{-F(\lambda_1)}}{1 + p_1 e^{\Delta F(\mu_1; \lambda_1)}}, \]

\[ \psi_1(\lambda) = e^{F(\lambda)} - \left( \frac{\lambda_1}{\lambda - \lambda_1} + \frac{\mu_1}{\lambda + \mu_1} \right) \frac{2i a e^{\Delta F(\mu_1; \lambda_1)} e^{F(\lambda)}}{1 + p_1 e^{\Delta F(\mu_1; \lambda_1)}}. \]

For the next, more complicated \([\text{with two terms } N=2 \text{ in } R_0]\) exact potential \( u_{N=2} \), one obtains via (32) and (45) after simple calculations the expression

\[ u_{N=2}(z, \bar{z}) = -\epsilon - 2\epsilon \frac{N(z, \bar{z})}{D(z, \bar{z})}, \]

here the nominator \( N \) and denominator \( D \) by the expressions

\[ \psi_{N=2}(z, \bar{z}) = \frac{e^{M(z, \bar{z})}}{1 + p_1 e^{\Delta F(\mu_1; \lambda_1)}}. \]
The formula for \( N(z, \bar{z}) \) under condition 
\[
N(z, \bar{z}) = \left( \lambda_1 - \mu_1 \right)^2 \frac{e^{\Delta F(\mu_2, \lambda_1)}}{\lambda_1 \mu_1} \left(q p_2 e^{2\Delta F(\mu_2, \lambda_2)} + p_1 \right) \]
\[
+ \left( \lambda_2 - \mu_2 \right)^2 \frac{e^{\Delta F(\mu_2, \lambda_2)}}{\lambda_2 \mu_2} \left(q p_1 e^{2\Delta F(\mu_1, \lambda_1)} + p_2 \right) \]
\[
+ p_1 p_2 (\lambda_1 - \mu_1 - \lambda_2 + \mu_2) \left( \frac{\lambda_1 - \mu_1}{\lambda_1 \mu_1} - \frac{\lambda_2 - \mu_2}{\lambda_2 \mu_2} \right) e^{\Delta F(\mu_1, \lambda_1) + \Delta F(\mu_2, \lambda_2)} \]
\[
+ q (\lambda_1 - \mu_1 + \lambda_2 - \mu_2) \left( \frac{\lambda_1 - \mu_1}{\lambda_1 \mu_1} + \frac{\lambda_2 - \mu_2}{\lambda_2 \mu_2} \right) e^{\Delta F(\mu_1, \lambda_1) + \Delta F(\mu_2, \lambda_2)}, \tag{54}
\]

\( D(z, \bar{z}) = (1 + p_1 e^{\Delta F(\mu_1, \lambda_1)} + p_2 e^{\Delta F(\mu_2, \lambda_2)} + q e^{\Delta F(\mu_1, \lambda_1) + \Delta F(\mu_2, \lambda_2)})^2 \) are given. It is remarkable that for the choice \( q=p_1p_2 \), i.e., under the condition
\[
\frac{(\lambda_1 - \lambda_2)(\lambda_2 + \mu_1)(\mu_1 - \mu_2)(\lambda_1 + \mu_2)}{(\lambda_1 + \lambda_2)(\lambda_2 - \mu_1)(\mu_1 + \mu_2)(\lambda_1 - \mu_2)} = -1, \tag{56}
\]

or equivalently under relation
\[
(\lambda_1 \mu_1 + \lambda_2 \mu_2)(\lambda_1 - \mu_1)(\lambda_2 - \mu_2) = 0, \tag{57}
\]

the formula for \( N=2 \) exact potentials \( u_{[N=2]} \) (53) with \( N, D \) given by (54) and (55) reduces to very simple expression,
\[
u(z, \bar{z}) = - \frac{e^{-2p_1(\mu_1 - \lambda_1)^2}}{\mu_1 \lambda_1} \frac{e^{\Delta F(\mu_1, \lambda_1)}}{(1 + p_1 e^{\Delta F(\mu_1, \lambda_1)})^2} \]
\[
- \frac{e^{-2p_2(\mu_2 - \lambda_2)^2}}{\mu_2 \lambda_2} \frac{e^{\Delta F(\mu_2, \lambda_2)}}{(1 + p_2 e^{\Delta F(\mu_2, \lambda_2)})^2}. \tag{58}
\]

Let us underline that in the present paper, exponentially localized exact potentials (line solitons) are considered, for such solutions by construction \( \mu_k \neq \lambda_k, \ (k=1, 2) \). By this reason, due to (57), the condition \( q=p_1p_2 \) satisfies if
\[
\lambda_1 \mu_1 + \lambda_2 \mu_2 = 0. \tag{59}
\]

Corresponding to \( u_{[N=2]} \) potentials (58), wave functions calculated in considered case by formulas (24) and (28), under condition \( p_1p_2=q \), have the following simple forms:
\[
\chi_2(\mu_1) = \overline{\chi_1} \chi_2 \left[ 1 - ia_2 e^{\Delta F(\mu_2, \lambda_2)} \left( \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2} \right) \right], \tag{60}
\]
\[
\chi_2(\lambda_1) = \overline{\chi_1} \chi_2 \left[ 1 - ia_2 e^{\Delta F(\mu_2, \lambda_2)} \left( \frac{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2} \right) \right], \tag{61}
\]
\[
\chi_2(\mu_2) = \overline{\chi_1} \chi_2 \left[ 1 + ia_1 e^{\Delta F(\mu_1, \lambda_1)} \left( \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2} \right) \right], \tag{62}
\]
\[
\chi_2(\lambda_2) = \overline{\chi_1} \chi_2 \left[ 1 + ia_1 e^{\Delta F(\mu_1, \lambda_1)} \left( \frac{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2} \right) \right], \tag{63}
\]
\[ \chi_2(\lambda) = 1 + 2i \left( \frac{\lambda_1 a_1}{\lambda_1 - \lambda} \chi_2(\mu_1) e^{\Delta F(\mu_1, \lambda_1)} + \frac{\mu_1 a_1}{\mu_1 - \lambda} \chi_2(-\lambda_1) e^{\Delta F(\mu_1, \lambda_1)} \right. \\
\left. + \frac{\lambda_2 a_2}{\lambda_2 - \lambda} \chi_2(\mu_2) e^{\Delta F(\mu_2, \lambda_2)} + \frac{\mu_2 a_2}{\mu_2 - \lambda} \chi_2(-\lambda_2) e^{\Delta F(\mu_2, \lambda_2)} \right), \quad (64) \]

Here \( \tilde{\chi}_1 \) and \( \tilde{\chi}_2 \) are the wave functions [see (49)]

\[ \tilde{\chi}_1 = \chi_1(\mu_1) = \chi_1(-\lambda_1) = \frac{1}{1 + p_1 e^{\Delta F(\mu_1, \lambda_1)}}, \]

\[ \tilde{\chi}_2 = \chi_1(\mu_2) = \chi_1(-\lambda_2) = \frac{1}{1 + p_2 e^{\Delta F(\mu_2, \lambda_2)}}, \quad (65) \]

corresponding to \( u_{N=1} \) potential. Corresponding to (60)–(64), wave functions \( \psi_2 \), given by formulas (30) and satisfying to linear auxiliary problem (3) and (4), have the following forms:

\[ \psi_2(\mu_1) = \frac{e^{F(\mu_1)}}{1 + p_1 e^{\Delta F(\mu_1, \lambda_1)}} \frac{1 + p_2 e^{\Delta F(\mu_2, \lambda_2)} (\lambda_1 + \lambda_2)(\lambda_2 + \mu_1)}{1 + p_2 e^{\Delta F(\mu_2, \lambda_2)}}, \quad (66) \]

\[ \psi_2(-\lambda_1) = \frac{e^{-F(-\lambda_1)}}{1 + p_1 e^{\Delta F(\mu_1, \lambda_1)}} \frac{1 + p_2 e^{\Delta F(\mu_2, \lambda_2)} (\lambda_1 - \lambda_2)(\lambda_2 + \mu_1)}{1 + p_2 e^{\Delta F(\mu_2, \lambda_2)}}, \quad (67) \]

\[ \psi_2(\mu_2) = \frac{e^{F(\mu_2)}}{1 + p_2 e^{\Delta F(\mu_2, \lambda_2)}} \frac{1 - p_1 e^{\Delta F(\mu_1, \lambda_1)}}{1 + p_1 e^{\Delta F(\mu_1, \lambda_1)}} \frac{1 + p_2 e^{\Delta F(\mu_2, \lambda_2)} (\lambda_1 + \lambda_2)(\lambda_2 - \mu_1)}{1 + p_2 e^{\Delta F(\mu_2, \lambda_2)}}, \quad (68) \]

\[ \psi_2(-\lambda_2) = \frac{e^{-F(-\lambda_2)}}{1 + p_2 e^{\Delta F(\mu_2, \lambda_2)}} \frac{1 - p_1 e^{\Delta F(\mu_1, \lambda_1)}}{1 + p_1 e^{\Delta F(\mu_1, \lambda_1)}} \frac{1 + p_2 e^{\Delta F(\mu_2, \lambda_2)} (\lambda_1 - \lambda_2)(\lambda_2 - \mu_1)}{1 + p_2 e^{\Delta F(\mu_2, \lambda_2)}}, \quad (69) \]

\[ \psi_2(\lambda) = e^{F(\lambda)} + 2i \left( \frac{\lambda_1 a_1}{\lambda_1 - \lambda} \psi_2(\mu_1) e^{-F(\lambda_1)} + \frac{\mu_1 a_1}{\mu_1 - \lambda} \psi_2(-\lambda_1) e^{F(\lambda_1)} \right. \\
\left. + \frac{\lambda_2 a_2}{\lambda_2 - \lambda} \psi_2(\mu_2) e^{-F(\lambda_2)} + \frac{\mu_2 a_2}{\mu_2 - \lambda} \psi_2(-\lambda_2) e^{F(\lambda_2)} \right) e^{F(\lambda)}, \quad (70) \]

Finally, in all derived, in the present section, formulas (44)–(70), reality condition (21) should be satisfied. The reality condition \( u = \bar{u} \) imposes additional restrictions on the parameters \( a_k, \lambda_k, \mu_k \) (43) of kernel (42). These restrictions and the calculations of exact potentials \( u \) with corresponding wave functions will be performed in Sec. IV.

### IV. EXACT LINE SOLITON POTENTIALS OF THE 2D SCHRODINGER EQUATION

For the case of complex space variables \( z = x + iy, \bar{z} = x - iy \), an application of reality condition (21) to each term of sum (42) for \( R_0 \) gives the following relation:
Let us underline that in the present paper, complex delta-functions are introduced. The last equality in (71) by the well known property of complex delta-functions \( \delta(\varphi(z)) = \sum_k \delta(z-z_k)/|\varphi'(z_k)|^2 \) is obtained, here \( z_k \) are simple roots of equation \( \varphi(z_k) = 0 \).

From (71) two possibilities follow:

\[
\begin{align*}
(1) & \quad a_k \lambda_k = \frac{\bar{a}_k}{\mu_k} \lambda_k = -\frac{\epsilon}{\mu_k}, \quad \mu_k = -\frac{\epsilon}{\lambda_k}, \\
(2) & \quad a_k \lambda_k = \frac{\bar{a}_k}{\lambda_k} \mu_k = \frac{\epsilon}{\lambda_k}, \quad \lambda_k = \frac{\epsilon}{\mu_k}.
\end{align*}
\]

For the first case in (72) taking into account the reality of \( \epsilon \), one obtains

\[
a_k = -\bar{a}_k := ia_{k0}, \quad \epsilon = -\mu_k \lambda_k = -\bar{\mu}_k \lambda_k, \quad \arg(\mu_k) = \arg(\lambda_k) + m \pi,
\]

i.e., pure imaginary amplitudes \( a_k (a_{k0}=\bar{a}_{k0}) \) and the relation between arguments of discrete spectral points \( \mu_k \) and \( \lambda_k \) with \( m \) arbitrary integer. From the second possibility in (72) for satisfying reality condition (21), the following relations:

\[
a_k = \bar{a}_k = a'_{k0}, \quad \epsilon = |\mu_k|^2 = |\lambda_k|^2, \quad \arg(\mu_k) = \arg(\lambda_k) + \delta_k
\]

with real amplitudes \( a_k = \bar{a}_k = a'_{k0} \) and arbitrary constants \( \delta_k \) follow.

So kernels (23) and (42) satisfying to potentiality (22) and reality (21) conditions in considered two cases due to (72)–(74) can be chosen in the following form:

\[
R_0(\mu, \bar{\mu}, \lambda, \bar{\lambda}) = \pi \sum_{k=1}^{2(L+N)} A_k \delta(\mu - M_k) \delta(\lambda - \Lambda_k)
\]

of \( L \) pairs of the type \( i \pi(a_{k0} \lambda_k \delta(\mu - \lambda_k) + a_{k0} \mu_k \delta(\mu - \lambda_k) + a_{k0} \mu_k \delta(\lambda - \mu_k)) \) (\( k = 1, \ldots, L \) \( \epsilon = -\lambda_k \mu_k \)) and \( N \) pairs of the type \( i \pi(a'_{k0} \lambda'_k \delta(\mu - \lambda'_k) + a'_{k0} \mu'_k \delta(\mu - \lambda'_k) + a'_{k0} \mu'_k \delta(\lambda - \mu'_k)) \) (\( k = 1, \ldots, N \)) (at \( \lambda_k = \lambda'_k \)). Here in (75) for application of general determinant formulas (27) and (32) due to (72)–(74), the following sets of amplitudes \( A_k \) and spectral parameters \( M_k, \Lambda_k \):

\[
\begin{align*}
(A_1, \ldots, A_{2(L+N)}) &= (i a_{10} \lambda_1, \ldots, i a_{10} \lambda_L, i a_{10} \mu_1, \ldots, i a_{10} \mu_L; a'_{10} \lambda'_1, \ldots, a'_{10} \lambda'_L, a'_{10} \mu'_1, \ldots, a'_{10} \mu'_L), \\
(M_1, \ldots, M_{2(L+N)}) &= (\mu_1, \ldots, \mu_L; -\lambda_1, \ldots, -\lambda_L; \mu'_1, \ldots, \mu'_L; -\lambda'_1, \ldots, -\lambda'_L)
\end{align*}
\]

are introduced.

General determinant formula (32) with matrix \( A \) from (27) with corresponding parameters (76) of kernel \( R_0 \) (75) [with \( 2(L+N) \) pairs of terms] of \( \tilde{\delta} \)-problem (6) gives exact \( [L,N] \) multiline potentials \( \varphi (z, \tilde{z}; t) \) with constant asymptotic value \( -\epsilon \) at infinity of linear problem (4). Simultaneously, an application of general scheme of the \( \tilde{\delta} \)-dressing method gives exact potentials \( V_{Schr} := -2\tilde{u} = -2u - 2\epsilon \) and corresponding wave functions \( \psi^{L,N}(M_1) = \chi^{L,N}(M_1)e^{F(M_1)} \) at discrete spectral parameters \( M_1 \) and \( \psi^{L,N}(\lambda) = \chi^{L,N}(\lambda)e^{F(\lambda)} \) at continuous spectral parameter \( \lambda \) of two-dimensional stationary Schrödinger equation (3) with fixed energy level \( E = -2\epsilon \).
All the rest of the present section will be devoted to the presentation for considered first case (72) of the explicit forms of exact one line soliton potentials of type [1,0] and exact two line soliton potentials of type [2,0] with corresponding wave functions for two-dimensional stationary Schrödinger equation (3).

To [1,0], [2,0] line potentials, the kernels of type $R_0$ (75) with values $L=1, 2; N=0$ (i.e., $a_{l0} \neq 0, l=1, 2; a'_{l0}=0, n=1, \ldots, N$) in (76) correspond. For nonsingular one line [1,0] and two line [2,0] potentials, parameters $\mu_k, \lambda_k, a_k$ in general formulas (44)–(65) of Sec. III must be identified due to (76) by the following way:

$$a_k = -\bar{a}_k := i a_{k0} \quad (k = 1, 2), \quad \epsilon = -\bar{\mu}_k \lambda_k = -\mu_k \bar{\lambda}_k \quad (k = 1, 2),$$

(77)

and real parameters $p_k$ (46),

$$p_k = \frac{a_{k0}(\lambda_k + \mu_k)}{\lambda_k - \mu_k} := e^{\phi_k} > 0$$

(78)
as positive constants must be chosen. Real phases $\Delta F(\mu_k, \lambda_k) = F(\mu_k) - F(\lambda_k) := \varphi_k (k = 1, 2)$ (44)–(65) are given in the considered case by the expressions

$$\varphi_k(z, \bar{z}) = i(1 + \mu_k - \lambda_k)z - (\bar{\mu}_k - \bar{\lambda}_k)\bar{z} = -2|\Delta| \left( \frac{\Delta_{1\mu}}{|\Delta|} x + \frac{\Delta_{1\lambda}}{|\Delta|} y \right),$$

(79)

where $\Delta_k = \Delta_{1\mu} + i \Delta_{1\lambda} = \mu_k - \lambda_k$.

The exact [1,0] soliton potential corresponding to the simplest kernel $R_0$ ($L=1, N=0$) of the type (75) with parameters (76) due to (77)–(79) is nonsingular one line soliton potential,

$$u = -\epsilon - \frac{e(\lambda_1 - \mu_1)^2}{2 \lambda_1 \mu_1} \frac{1}{\cosh \frac{1}{2} \varphi_1 + \phi_01} = -\epsilon + \frac{|\lambda_1 - \mu_1|^2}{2 \cosh \frac{1}{2} \varphi_1 + \phi_01}, \quad \epsilon = -\lambda_1 \bar{\mu}_1.$$  

(80)

Maximum value of $u$ (or minimum of $V_{Schr} = -2\bar{u}$) achieves along the line $\varphi_1 + \phi_01 = -2(\Delta_{1\mu} + \Delta_{1\lambda}) + \phi_01 = 0$ and the exact potential is invariant under translations along this line; due to this fact, it is customary to use for such potential the term—one line soliton potential. The corresponding wave functions $\psi^{[1,0]}(\mu_1) = \chi^{[1,0]}(\mu_1) e^{F(\mu)}, \psi^{[1,0]}(-\lambda_1) = \chi^{[1,0]}(-\lambda_1) e^{F(-\lambda_1)}$, and $\psi^{[1,0]}(\lambda) = \chi^{[1,0]}(\lambda) e^{F(\lambda)}$ of linear auxiliary problem (4) and exact potential $V_{Schr} = -2\bar{u}$ of 2D stationary Schrödinger equation (3) with energy level $E = -2\epsilon$ due to (3), (30), (49), and (50) have the following forms:

$$\psi(\mu_1) = e^{F(\mu)} + \frac{e^{i(\lambda_1 - \mu_1)^2 - |\mu_1|^2}}{e^{F(\mu)} + 1} \frac{\Delta_{1\mu}}{|\Delta|} \frac{\Delta_{1L}}{|\Delta|} \cosh \frac{1}{2},$$

(81)

$$\psi(-\lambda_1) = e^{F(-\lambda)} = e^{-i(\lambda_1 - |\mu_1|^2 - \lambda_1^2)} \frac{\Delta_{1\mu}}{|\Delta|} \frac{\Delta_{1L}}{|\Delta|} \cosh \frac{1}{2},$$

(82)

$$\psi(\lambda) = e^{F(\lambda)} + \frac{\lambda_1}{\lambda - \lambda_1} + \frac{\mu_1}{\lambda + \mu_1} \frac{2a_{k0} e^{2\psi} + F(\lambda)}{e^{2\psi} + \phi_01},$$

(83)
FIG. 1. Potential $V_{Schr}$ (84) (dark) with the energy level $E$ (dark plane) and corresponding squared absolute values of wave functions $|\psi(0,1)|^2=|\psi(0,-1)|^2$ (81) and (82) (light) with parameters (a) $a_{10}=-0.1, \lambda_1=e^{i\pi}, \mu_1=4e^{i\pi}, E=-2\epsilon=-8$, (b) $a_{10}=-0.1, \lambda_1=e^{i\pi}, \mu_1=4e^{i\pi}, E=-2\epsilon=8$, and (c) $a_{10}=0.1, \lambda_1=e^{i\pi}, \mu_1=0, E=-2\epsilon=0$.

The corresponding to exact two line soliton potential in considered case of kernel $R_0$ of type (75) with parameters (76) is given by formula (53). It is remarkable that under the condition $q=p_1p_2$, this potential radically simplifies. Indeed, due to (59), condition $q=p_1p_2$ is satisfied if $\lambda_1\mu_1+\lambda_2\mu_2=0$, in this case exact two line soliton potential $V_{Schr}=-2\mu=-2u+2\epsilon$ with $u$ given by (53) takes the form

$$V_{Schr} = -\frac{E(\lambda_1 - \mu_1)^2}{\lambda_1\mu_1} \frac{1}{\cosh^2 \varphi_1 + \phi_0^2} = -\frac{|\lambda_1 - \mu_1|^2}{2 \cosh^2 \varphi_1 + \phi_0^2}; \quad E = -2\epsilon = 2\lambda_1\mu_1.$$  \hspace{2cm} (84)

Graphs of exact one line $[1,0]$ potential (84) and squared absolute value of wave functions (81) and (82) for some values of corresponding parameters on Fig. 1 are shown.

Exact two line $[2,0]$ soliton potential in considered case of kernel $R_0$ of type (75) with parameters (76) is given by formula (53). It is remarkable that under the condition $q=p_1p_2$, this potential radically simplifies. Indeed, due to (59), condition $q=p_1p_2$ is satisfied if $\lambda_1\mu_1+\lambda_2\mu_2=0$, in this case exact two line soliton potential $V_{Schr}=-2\mu=-2u+2\epsilon$ with $u$ given by (53) takes the form

$$V_{Schr} = -\frac{\epsilon(\lambda_1 - \mu_1)^2}{\lambda_1\mu_1} \frac{1}{\cosh^2 \varphi_1(z,\bar{z}) + \phi_0^2} + \frac{\epsilon(\lambda_2 - \mu_2)^2}{\lambda_2\mu_2} \frac{1}{\cosh^2 \varphi_2(z,\bar{z}) + \phi_0^2}$$
$$= -\frac{|\lambda_1 - \mu_1|^2}{2 \cosh^2 \varphi_1(z,\bar{z}) + \phi_0^2} - \frac{|\lambda_2 - \mu_2|^2}{2 \cosh^2 \varphi_2(z,\bar{z}) + \phi_0^2}.$$  \hspace{2cm} (85)

From the relation $\lambda_1\mu_1+\lambda_2\mu_2=0$, taking into account first condition (72) ($\lambda_1\bar{\mu}_1=\lambda_1\mu_1=\lambda_2\mu_2=\lambda_2\bar{\mu}_2=-\epsilon$) follows $\bar{\mu}_2/\bar{\mu}_1=-\mu_2/\mu_1$, and from the last relation, one obtains

$$\mu_2 = i\sigma\mu_1, \quad \lambda_2 = i\sigma^{-1}\lambda_1, \quad \sigma = \bar{\sigma}.$$  \hspace{2cm} (86)

The corresponding to exact two line $[2,0]$ soliton potential (85) wave functions in considered case of kernel $R_0$ of type (75) with parameters (76)–(79), under condition $p_1p_2=q$ are given by very simple expressions (60)–(64). Graphs of two line $[2,0]$ exact potential $V_{Schr}=-2\mu=-2u+2\epsilon$
corresponding wave functions [81] and (82) for some values of corresponding parameters on Figs. 2 and 3 are shown.

General case of multiline \([N,L]\) soliton potentials with corresponding wave functions in the context of NVN equation will be considered elsewhere.

![Figure 2](image1.png)

**FIG. 2.** Potential \(V_{\text{Schr}} (85)\) for the energy level \(E\) (dark plane) with parameters (a) \(a_{10}=-1, \lambda_1=e^{i\pi/8}, \mu_1=1.05e^{i\pi/8}; a_{20}=-1, \sigma=1, E=-2e^{-1.1}\), (b) \(a_{10}=-0.1, \lambda_1=e^{i\pi/8}, \mu_1=4e^{i\pi/8}; a_{20}=-0.1, \sigma=1, E=-2e=8,\) and (c) \(a_{10}=0.1, \lambda_1=e^{i\pi/8}, \mu_1=0; a_{20}=0.1, \sigma=1, E=-2e=0.\)

![Figure 3](image2.png)

**FIG. 3.** Squared absolute values of wave functions \(|\phi^{(2,0)}(\mu_1)|^2=|\phi^{(2,0)}(-\lambda_1)|^2\) (light) corresponding to different values of energy \(E\) in the Figs. 2(a)–2(c).
V. PHYSICAL PROPERTIES OF STATIONARY STATES OF QUANTUM PARTICLE IN MULTILINE SOLITON POTENTIALS OF 2D SCHRÖDINGER EQUATION

At present section, physical properties of stationary states of quantum particle with unit mass \( m = 1 \) moving in field of exact one line or two line soliton potentials calculated in preceding section will be discussed. Due to (79), the unit vector \( \hat{l} \) along the line of minimum value one line potential (84) and projection of quantum mechanical momentum operator of particle on this line are given by expressions

\[
\hat{\mathbf{l}} = \left( -\frac{\Delta_{1R}}{|\Delta_1|}, \frac{\Delta_{1L}}{|\Delta_1|} \right), \quad \hat{\mathbf{p}}_l := -i\hat{\mathbf{l}} \cdot \hat{\nabla} = i \frac{\Delta_{1L}}{|\Delta_1|} (\hat{\partial}_1 \partial_x + \Delta_{1R} \partial_y),
\]

(87)

for convenience here and below, we use system of units with unit Plank constant \( \hbar = 1 \) and all space derivatives in terms of complex derivatives \( \hat{\partial}_1 := \frac{1}{2}(\partial_x - i\partial_y) \), \( \hat{\partial}_2 := \frac{1}{2}(\partial_x + i\partial_y) \) are expressed.

The one line potential wave functions \( \psi(\mu_1) \) and \( \psi(-\lambda_1) \) due to (81), (83), and (87) evidently are eigenfunctions of operator of momentum \( \hat{p}_l \),

\[
\hat{p}_l \psi(\mu_1) = \left| \mu_1 \right|^2 - \left| \mu_1 \right|^2 \psi(\mu_1), \quad \hat{p}_l \psi(-\lambda_1) = -\left| \lambda_1 \right|^2 - \left| \mu_1 \right|^2 \psi(-\lambda_1),
\]

(88)

with the projections \( p_l \) of momentum of particle on the direction \( \hat{l} \) given by \( \pm p_l = \pm (|\mu_1|^2 - |\mu_1|^2)/|\Delta_1| \). "Longitudinal" energy of particle corresponding to its motion along the line of minimum value of one line potential (84) due to \( \epsilon = -\mu_1 \bar{\mu}_1 = -\bar{\mu}_1 \mu_1 \) and (88) has the following value:

\[
E_l = p_l^2 = \frac{\mu_1^2}{2m} = \frac{|\mu_1 + \lambda_1|^2}{2|\mu_1 - \lambda_1|^2} = \frac{|\mu_1 + \lambda_1|^2}{2}.
\]

(89)

Energy of particle motion in transverse to one line soliton potential (84) valley is the difference between its total \( E = -2\epsilon = \mu_1 \bar{\mu}_1 + \bar{\mu}_1 \mu_1 \) and longitudinal \( E_l \) energies,

\[
E_{\text{tr}} = E - E_l = \mu_1 \bar{\mu}_1 + \bar{\mu}_1 \mu_1 - \frac{|\mu_1 + \lambda_1|^2}{2} = -\frac{\left| \mu_1 - \lambda_1 \right|^2}{2}.
\]

(90)

It is interesting to note that in Cartesian system of coordinates,

\[
\chi' := \frac{\Delta_{1L}}{|\Delta_1|} x + \frac{\Delta_{1R}}{|\Delta_1|} y, \quad y' := -\frac{\Delta_{1R}}{|\Delta_1|} x + \frac{\Delta_{1L}}{|\Delta_1|} y,
\]

(91)

with axes \( \chi' \) and \( y' \) chosen in transverse and longitudinal to one line soliton potential (84) directions 2D stationary Schrödinger equation for the wave functions \( \psi(\mu_1) \) or \( \psi(-\lambda_1) \) with energy \( E = \mu_1 \bar{\mu}_1 + \bar{\mu}_1 \mu_1 = E_{\text{tr}} + E_l \) due to (81), (84), and (91) takes the form

\[
\left( -\frac{1}{2} \frac{\Delta_{1L}^2}{\cosh^2\left( |\Delta_1| x' - \frac{\phi_{01}}{2} \right)} \right) \frac{\partial^2}{\partial x'^2} + \frac{\cosh^2\left( |\Delta_1| x' - \frac{\phi_{01}}{2} \right)}{\cosh\left( |\Delta_1| y' - \frac{\phi_{01}}{2} \right) \cosh\left( |\Delta_1| x' - \frac{\phi_{01}}{2} \right)} = E \cdot e^{\frac{\phi_{01}}{2}} e^{\frac{\phi_{01}}{2}}.
\]

(92)

Last equation via the separation of variables reduces in two one-dimensional (1D) stationary Schrödinger equations,

\[
\left( -\frac{1}{2} \frac{\Delta_{1L}^2}{\cosh^2\left( |\Delta_1| x' - \frac{\phi_{01}}{2} \right)} \right) \frac{1}{\cosh\left( |\Delta_1| x' - \frac{\phi_{01}}{2} \right)} = \frac{E_{\text{tr}}}{\cosh\left( |\Delta_1| x' - \frac{\phi_{01}}{2} \right)},
\]

(92)
with corresponding energies $E_u = -|\mu_1 - \lambda_1|^2 / 2$, $E_t = |\mu_1 + \lambda_1|^2 / 2$ of transverse and longitudinal motions of particle. Due to performed analysis, one can assert that the wave functions $\psi(\mu_1)$ and $\psi(-\lambda_1)$ correspond to stationary states of particle with total energy $E = E_t + E_u$ and longitudinal momentum values $\pm p_t = \pm (|\lambda_1|^2 - |\mu_1|^2) / |\Delta_1|$. Particle is bounded by one line potential (84) in direction of axis $x'$ (transverse to potential valley) and moves freely in direction of axis $y'$ (along to potential valley); all these illustrate graphs plotted on Fig. 1. The total energy can be chosen to be positive and as large as possible but the particle remains to be bounded in transverse to potential valley direction: the increase of longitudinal energy $E_t = |\mu_1 + \lambda_1|^2 / 2$ means simultaneously lowering of the depth $V_{\text{min}} = -|\lambda_1|^2 = -|\mu_1 - \lambda_1|^2 = 2E_u$ of soliton potential well and transverse energy $E_u$ of particle. Spectral parameters $\mu_1, \lambda_1$ connected with fixed total energy by the relation $E = -2e = \mu_1 \lambda_1 + \mu_2 \lambda_2$, one can use as regulating behavior of particle in potential line.

The wave functions of particle in two line soliton potential (85) are given by expressions (66)–(69). For simplicity, it will be considered at first their zero energy limit: $E = -2e = \mu_2 \lambda_1 + \mu_2 \lambda_2 \rightarrow 0, (k = 1, 2)$. Such limit can be correctly performed by the following settings in all corresponding formulas: $e \rightarrow 0$ and $\mu_k \rightarrow 0$ in cases when uncertainty is absent, but $e / \mu_k \rightarrow \mu_2 / \lambda_k$ in accordance with the relation $e = -\mu_2 \lambda_k$, in addition the formula $\lambda_k = \pm \sigma^2 \lambda_k$ with arbitrary real constant $\sigma$ [following from the relations $\mu_k \lambda_k = \mu_2 \lambda_k$ and $\mu_1 \lambda_1 + \mu_2 \lambda_2 = 0$, see (86)] is assumed to be valid. The two line potential due to (80) has the form

$$V_{\text{Sch}} = -\frac{|\lambda_1|^2}{\cosh^2 \varphi_1(z, \bar{z}) + \phi_{01}^2} - \frac{|\lambda_2|^2}{\cosh^2 \varphi_2(z, \bar{z}) + \phi_{02}^2}.$$  \hspace{1cm} (94)

here the phases $F(\mu_1), F(\lambda_1), \varphi(z, \bar{z})$ due to (8), (79), and (91) have in considered limit the forms

$$F(\mu_1) = -\overline{F(\lambda_1)} = i |\lambda_1| y'-|\lambda_1| x', \quad \varphi_1(z, \bar{z}) = \Delta F(\mu_1, \lambda_1) = -2 |\lambda_1| x', \quad \varphi_2(z, \bar{z}) = \Delta F(\mu_2, \lambda_2) = 2 |\lambda_1| y' / \sigma.$$ \hspace{1cm} (95)

The wave functions $\psi(\mu_1) = \psi_{\pm p_t}$ and $\psi(-\lambda_1) = \psi_{-p_t}$, with projections $\pm p_t$ on the direction $\tilde{t}$ (87) due to (66), (67), and (95) are given in the limit $e \rightarrow 0, \mu_k \rightarrow 0$ by the following expressions:

$$\psi_{\pm p_t} = \frac{1}{\cosh \left( |\lambda_1| x' - \phi_{01}^2 / 2 \right)} e^{\pm i |\lambda_1| y' (\sigma + i \tanh (|\lambda_1| y' / \sigma + \phi_{02} / 2))},$$ \hspace{1cm} (96)

here and below unessential constant multipliers of wave functions are omitted. Due to the $\phi$-dressing constructions, these functions satisfy to 2D stationary Schrödinger equation with zero energy $E = E_u + E_t = -|\lambda_1|^2 / 2 + |\lambda_1|^2 / 2 = 0$,

$$\left( -\frac{1}{2} \left( \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} \right) - \frac{|\lambda_1|^2}{\cosh^2 \left( |\lambda_1| x' - \phi_{01}^2 / 2 \right)} - \frac{|\lambda_1|^2}{\sigma^2} \frac{1}{\cosh^2 \left( |\lambda_1| y' / \sigma + \phi_{02}^2 / 2 \right)} \right) \times \frac{e^{\pm i |\lambda_1| y' (\sigma + i \tanh (|\lambda_1| y' / \sigma + \phi_{02} / 2))}}{\cosh \left( |\lambda_1| x' - \phi_{01}^2 / 2 \right)} = 0.$$ \hspace{1cm} (97)

Again the last equation can be separated in two 1D stationary Schrödinger equations with corresponding energies,
Due to (98), particle is bounded by one line potential \( V_{Schr}^{(1)} = -|\lambda_1|^2 \cosh^{-2}(|\lambda_1|x' - \frac{\phi_{01}}{2}) \) in direction of axis \( x' \). The states of infinite motion of particle in direction of axis \( y' \) in the field of potential \( V_{Schr}^{(2)} = -|\lambda_1|^2 \sigma^{-2} \cosh^{-2}(|\lambda_1|y'/\sigma + \frac{\phi_{02}}{2}) \) due to (99) are described by “deformed” plane waves \( e^{\pm i|\lambda_1|y'}(\sigma + i \tanh(|\lambda_1|y'/\sigma + \phi_{02}/2)) \). The localization of particle in stationary states with wave functions \( \psi_\mu, \psi_{-\mu} \) illustrates by Fig. 3(e).

Similar calculations one can perform are also in the case of nonzero values of energy \( E = \lambda_1 \mu_1 + \lambda_1 \mu_2 = -\epsilon \neq 0 \). The phases \( F(\mu_1), F(\lambda_1), \varphi_\mu(z, \bar{z}) \) due to (8), (79), and (91) have, in considered case, the forms

\[
F(\mu_1) = -\overline{F(\lambda_1)} = -\frac{|\Delta_1| x' + i y'}{|\Delta_1|} \frac{|\lambda_1|^2 - |\mu_1|^2}{2}, \quad \varphi_\mu(z, \bar{z}) = \Delta F(\mu_1, \lambda_1) = -2|\Delta_1|x',
\]

\[
\varphi_{\pm 1}(z, \bar{z}) = \Delta F(\mu_2, \lambda_2) = 2y' \frac{(\mu_1 - \lambda_1)(\sigma \mu_1 - \sigma^{-1} \lambda_1)}{|\Delta_1|}.
\]

(100)

The wave functions \( \psi_\mu, \psi_{-\mu} \) and \( \psi_{-\mu}, \psi_\mu \) with projections \( \pm \mu_1 \) on the direction \( \bar{t} \) (87) due to (66), (67), (88), and (95) are given by the following expressions:

\[
\psi_{\pm \mu_1} = \frac{1}{\cosh(|\Delta_1|x' - \frac{\phi_{01}}{2})} e^{\pm i|\Delta_1|x'} \left( 1 - i \frac{(\sigma \mu_1 - \sigma^{-1} \lambda_1)}{\mu_1 + \lambda_1} \right) \times \tanh \left( \frac{(\mu_1 - \lambda_1)(\sigma \mu_1 - \sigma^{-1} \lambda_1)}{|\Delta_1|} y' + \frac{\phi_{02}}{2} \right),
\]

(101)

in the last formula unessential constant multipliers of wave functions are omitted. Due to the \( \bar{d} \)-dressing constructions, these functions satisfy to 2D stationary Schrödinger equation with energy \( E = -2\epsilon \).

\[
\left( -\frac{1}{2} \begin{pmatrix} \partial_x^2 & \partial_y^2 \\
\end{pmatrix} - \frac{|\Delta_1|^2}{\cosh^2(|\Delta_1|x' - \frac{\phi_{01}}{2})} - \frac{|\Delta_2|^2}{\cosh^2(|\mu_1 - \lambda_1)(\sigma \mu_1 - \sigma^{-1} \lambda_1)} \right) \psi(\mu_1) = E \psi(\mu_1).
\]

(102)

Again the last equation can be separated in two 1D stationary Schrödinger equations with corresponding energies. The equation for transverse motion of particle relative to minimum of potential \( V_{Schr}^{(1)} = -|\Delta_1|^2 \cosh^2(|\Delta_1|x' - \phi_{01}/2) \) has the form
The equations for the wave functions corresponding to longitudinal motion (along the minimum value of line potential $V_{Schr}^{(1)}$ of particle in the field of the soliton potential $V_{Schr}^{(2)} = -|\Delta x|^2/cosh^2\left(\frac{\mu_1 - \lambda_1}{|\Delta_1|}\right)$) are the following:

$$
\begin{align*}
-\frac{1}{2} \frac{\partial^2}{\partial x'^2} - \frac{|\Delta_1|^2}{\cosh^2\left(\frac{|\Delta_1| |x' - \phi_{01/2}|}{2}\right)} \frac{1}{\cosh\left(\frac{|\Delta_1| |x' - \phi_{01/2}|}{2}\right)} = -\frac{|\Delta_1|^2}{2} \frac{1}{\cosh\left(\frac{|\Delta_1| |x' - \phi_{01/2}|}{2}\right)}.
\end{align*}
$$

(103)

Due to these facts, one can conclude that the basis wave functions $\psi(\mu_1)$ (66), $\psi(-\lambda_1)$ (67) correspond to particle which is localized in transverse direction to the minimum of potential valley $V_{Schr}^{(1)}$ (see Fig. 3), the dimension of area of localization is given by $|\Delta x| \sim |\Delta_1|^{-1}$. Quite analogously, it can be shown that the wave functions $\psi(\mu_2)$ (68), $\psi(-\lambda_2)$ (69) at another discrete values of spectral parameters correspond to particle localized in transverse direction to the minimum of another potential valley $V_{Schr}^{(2)}$.

The $\tilde{\Delta}$-dressing method allows also to construct basis wave function $\psi(\lambda)$ (83) at continuous values of spectral parameter $\lambda$. Exponential multiplier $e^{F(\lambda)}$ of this function due to (8) has the form

$$
\begin{align*}
e^{F(\lambda)} = \exp\left[i\lambda_R \left(1 - \frac{e}{|\lambda|^2}\right) x - i\lambda_I \left(1 - \frac{e}{|\lambda|^2}\right) y - \lambda_R \left(1 + \frac{e}{|\lambda|^2}\right) y - \lambda_I \left(1 + \frac{e}{|\lambda|^2}\right) x\right]
\end{align*}
$$

(105)

and is bounded under condition $E = -2e = 2|\lambda|^2 > 0$. One can show that stationary states of quantum particle with such wave functions $\psi(\lambda)$ correspond to reflectionless motion of particle across one line soliton (84) and two line soliton (94) potentials. Here it will be demonstrated for one line soliton case. On Fig. 4 squared value of wave function (83) for particle in one line soliton potential well with some values of corresponding parameters are shown. Similar calculations are valid also for two line soliton case.

The density of probability current $\vec{j}$ for particle in state with wave function $\psi(\lambda)$ is defined by formula (here and below we use system of units in which $\hbar = 1$ and $m = 1$),

$$
\vec{j} := Im(\bar{\psi}(\lambda) \vec{\nabla} \psi(\lambda)),
$$

(106)

where $\bar{\psi}(\lambda)$ is complex conjugate to wave function $\psi(\lambda)$ and $\vec{\nabla}$-gradient operator.

The considered wave function is given by expression (83), where $F(\lambda)$ according to (105) and condition $e = -|\lambda|^2$ takes the form

$$
F(\lambda) = 2i(\lambda_R x + \lambda_I y) = i\vec{k} \cdot \vec{r},
$$

(107)

where vectors of position and momentum of particle are given by the formulas $\vec{r} = (x, y)$ and $\vec{k} = (2\lambda_R, 2\lambda_I)$. Substituting (83) in (106), after simple calculation, according to (107), one can obtain for density of probability current the following expression:
nient to use the coordinates

where, according to (79), $\nabla \varphi(z, \bar{z}, t) = 2(\lambda_{11} - \mu_{11}, \lambda_{1R} - \mu_{1R})$. Analogous expression can be derived for wave function $\hat{\tilde{p}} \psi(\lambda)$,

$$\hat{\tilde{p}} \psi(\lambda) = \hat{k} \psi(\lambda) - \left( \frac{\lambda_{11}}{\lambda - \lambda_{11}} + \frac{\mu_{11}}{\lambda + \mu_{11}} \right) \frac{2a_1 e^{[p, \bar{p}, t]}}{(1 + e^{[p, \bar{p}, t] + \phi_0})^2} \nabla \varphi(z, \bar{z}, t) e^{F(\lambda)}.$$  \hspace{1cm} (109)

The second terms in formulas (108) and (109) evidently disappear outside potential valley. The wave function $\psi(\lambda)$ outside potential well corresponds to the state of particle with momentum $\hat{k}$ (109); density of probability current (108) in turn is defined by the first term. So for particle moving across line soliton potential well (or scattering on this well), densities of probability current and momentum for this particle before well and after well are not changed and reflection current of probability is absent. This fact of absence of reflected wave during scattering means that considered one line soliton potential (84) is a reflectionless potential. Similar consideration applies to multilne soliton potentials, which are also reflectionless.

In conclusion of present section, the question about possibility of transitions between different stationary states with wave functions $\psi(\lambda)$ (83) and $\psi(\mu)$ (81), $\psi(-\lambda)$ (82) of quantum particle in one line soliton potential (84) will be considered. It is assumed that all these states belong to the same positive energy of particle $E = 2|\lambda_1|^2 = 2|\lambda_1 \mu_1| > 0$; the wave function $\psi(\lambda)$ corresponds to purely continuous spectrum, but the wave functions $\psi(\mu)$, $\psi(-\lambda)$ describe a quantum particle localized in transverse direction (along axis $\lambda'$) to the valley of potential $V_{Schr}(84)$. It is convenient to use the coordinates $x'$, $y'$ given by (91), in such coordinates, relation (107) takes the form

$$F(\lambda) = \hat{k} \cdot \vec{r}, \quad \hat{k} = 2 \left( \frac{\Delta_{1I}}{|\Delta_1|} + \frac{\Delta_{1R}}{|\Delta_1|} \right) \frac{\Delta_{1I}}{|\Delta_1|} - \frac{\Delta_{1R}}{|\Delta_1|}.$$

To basis stationary states of particle with energy $E = 2\lambda_1 \mu_1$ and localized across potential $V_{Schr}(84)$ correspond orthogonal to each other wave functions $\psi(\mu)$ and $\psi(-\lambda)$ given by (81) and (82).
\[
\int \int \overline{\psi(\mu_i)} \psi(-\lambda_i) dx' dy' = \frac{\pi}{|\Delta_i|} \delta(p_i - (-p_i)) = 0, \tag{111}
\]

where \(p_1 = (|\lambda_1|^2 - |\mu_1|^2)/|\Delta_1|\) and \(-p_1 = -(|\lambda_1|^2 - |\mu_1|^2)/|\Delta_1|\) are projections of operator of momentum of particle on \(y'\) axis corresponding to stationary states with wave functions \(\psi(\mu_i)\) and \(\psi(-\lambda_i)\).

By the same manner, one can easily calculate the scalar products of wave function \(\psi(\lambda)\) (83) of pure continuous spectrum with wave functions \(\psi(\mu_i)\) (81) and \(\psi(-\lambda_i)\) (82) of stationary states of particle localized along potential valley; remember that due to constructions of the \(\bar{\partial}\)-dressing method in the present paper, all mentioned states correspond to fixed energy \(E = 2|\lambda|^2 = 2\lambda_1\mu_1 > 0\). These scalar products in well known terminology of quantum mechanics represent amplitudes of probabilities of transitions between corresponding stationary states and are given by the following expressions:

\[
\int \int \overline{\psi(\mu_1)} \psi(\lambda) dx' dy' = \frac{\pi^2 e^{-ik'y/|\Delta|}}{e^{\phi_0/2}|\Delta|} \text{sech}\left(\frac{\pi k^2}{2|\Delta|}\right) \left[1 + \frac{1 - ik'y/|\Delta|}{2e^{\phi_0/2}}\right] \delta(k'_y - p_1), \tag{112}
\]

\[
\int \int \overline{\psi(-\lambda_i)} \psi(\lambda) dx' dy' = \frac{\pi^2 e^{-ik'y/|\Delta|}}{e^{\phi_0/2}|\Delta|} \text{sech}\left(\frac{\pi k^2}{2|\Delta|}\right) \left[1 + \frac{1 - ik'y/|\Delta|}{2e^{\phi_0/2}}\right] \delta(k'_y - (-p_i)), \tag{113}
\]

where \(\alpha = \frac{\lambda_1 - \lambda_1^*}{\lambda_1 + \lambda_1^*}\).

The probabilities of considered transitions are proportional to squared absolute values of corresponding amplitudes (112) and (113). Per unit of length of potential (84), along potential valley, one obtains for such probabilities the following expressions:

\[
w_{\psi(\lambda) \rightarrow \psi(\mu_1)} = \frac{\pi^3}{e^{\phi_0}|\Delta|^2} \text{sech}^2\left(\frac{\pi k^2}{2|\Delta|}\right) \left[1 + \frac{1 - ik'y/|\Delta|}{2e^{\phi_0/2}}\right]^2 \delta(k'_y - p_1), \tag{114}
\]

\[
w_{\psi(\lambda) \rightarrow \psi(-\lambda_i)} = \frac{\pi^3}{e^{\phi_0}|\Delta|^2} \text{sech}^2\left(\frac{\pi k^2}{2|\Delta|}\right) \left[1 + \frac{1 - ik'y/|\Delta|}{2e^{\phi_0/2}}\right]^2 \delta(k'_y - (-p_i)). \tag{115}
\]

At first sight it seems from calculated expressions for probabilities (114) and (115) that particle in stationary state with wave function \(\psi(\lambda)\) (83) can be captured by soliton potential well (84) in cases for which “resonance conditions” are fulfilled: the projection of momentum \(k_y'\) of particle on the direction of minimum line (axis \(y'\)) of this potential must coincide with projections of momentum \(\pm p_1\) of particle in stationary states with wave functions \(\psi(\mu_1)\) (81) and \(\psi(-\lambda_i)\) (82); in accordance with uncertainty relation \(\Delta x \Delta k_x \approx 1\), probabilities (114) and (115) have nonzero values for following transverse to potential valley momenta of particle \(|k| \approx |\Delta| = |\mu_1 - \lambda_1|\).

But as was mentioned by Grinevich (in conversations of obtained results with the first author of the present paper in Workshop “Nonlinear Physics. Theory and experiment. VI,” Gallipoli, Italy, 23 June–3 July, 2010) that the last assertion is wrong: one can easily show that from definition (110) and due to equality \(E = 2|\lambda|^2 = 2\lambda_1\mu_1 > 0\), it follows that \(k_y' > p_1\), so the corresponding probabilities (114) and (115) are equal to zero. This confirms from yet another side that the stationary state of pure continuous spectrum \(E = 2|\lambda|^2\) with wave function \(\psi(\lambda)\) (83) corresponds to reflectionless motion of a particle in the considered one line soliton potential.

**VI. CONCLUSIONS**

The powerful \(\bar{\partial}\)-dressing method of Zakharov and Manakov, discovered a quarter of century ago, continues to apply successfully for constructions of exact solutions of multidimensional integrable nonlinear equations. The realization of the method goes due to basic idea of IST...
through the careful study of auxiliary linear problems by the methods of modern theory of functions of complex variables. By this way, one constructs exact complex wave functions (with rich analytical structure) of linear auxiliary problems and through the wave functions, via reconstruction formulas, exact transparent potentials of linear auxiliary problems and simultaneously exact solutions of integrable nonlinear equations. The exact transparent multilinear soliton potentials and wave functions of corresponding stationary states of 2D stationary Schrödinger equation (3) constructed in this paper with their remarkable physical interpretation may find an applications in solid state physics of planar nanostructures.

In conclusion, we would like to thank the first referee for his useful remarks to our paper.

ACKNOWLEDGMENTS

This research work is supported by scientific Grant of fundamental researches of Novosibirsk State Technical University (2010), by the Grant (Grant No. 2.1.1/1958) of Ministry of Science and Education of Russia Federation via analytical departmental special program “Development of potential of High School” (2009-2010), and by the Russia Fund of Basic Research Grant RFBR No. 09-01-92442-Kea (2009-2010).

23 Manakov, S. V., “The inverse scattering transform for the time-dependent Schrödinger equation and Kadomts-